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LETTER TO THE EDITOR

Universal connections, gauge anomalies and Lie group cohomology

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Abstract. The geometric interpretation, due to Thierry-Mieg, of the ghost field as certain components of an Ehresmann connection on the total space of a principal bundle, and of the BRS transformation as part of the exterior derivative on the total space, is discussed and amended to incorporate the spacetime dependence of the ghost field. Using this interpretation it is shown that non-trivial solutions to the Wess-Zumino consistency condition for gauge anomalies are related to non-trivial de Rham cohomology of the Lie group.

The algebraic approach to the study of anomalies (Baulieu 1984, Bonora and Cotta-Ramusino 1983, Stora 1983, Zumino 1983) has proved most successful in obtaining the functional form of non-Abelian gauge anomalies (up to a normalisation constant) by solving a certain linear condition, which anomalies must satisfy, known as the Wess-Zumino consistency condition. The solution makes use of invariant polynomial expressions in the fields, of form degree two greater than the dimensionality of spacetime, and the anomalies are obtained by analysing the algebraic properties of a double chain complex, consisting of polynomials in the Yang-Mills potential and field strength, A and F, the ghost field χ and its exterior derivative, operated on by the exterior derivative d and the BRS operator s. See Dubois-Violette *et al* (1985a, b) for a thorough analysis of the algebra involved. The cohomological meaning of this approach was clarified by Bonora and Cotta-Ramusino (1983), who introduced the notion of 'local cohomology': the (integrated) anomaly is a local expression in the fields, but arises as s of a non-local expression. Because of this, s of the anomaly is zero (the Wess-Zumino condition) but, on the other hand, the anomaly is essentially trivial if it may be written as s of a local expression. Non-trivial anomalies are thus non-trivial 'local' s-cocycles.

The geometric interpretation of these algebraic manipulations has been less clear. The first problem is to explain where the extra two dimensions, needed to define the invariant polynomial used at the start of the programme, come from and why the final expression for the anomaly has no support in these extra dimensions. The most convincing approach to this problem is to consider forms defined on the total space of an appropriate fibre bundle, associated with the Yang-Mills theory under consideration, rather than on spacetime itself, being the base space of the fibre bundle (Thierry-Mieg 1980a, b, Cotta-Ramusino 1985). The second problem is to clarify the nature of the ghost field and the BRS operator. In the Thierry-Mieg (1980a, b) approach the ghost is interpreted as the vertical component of the full connection (i.e. the Ehresmann connection, which is a form on the total space of the bundle—see below), whilst the BRS operator is the vertical part of the exterior derivative. This approach has been criticised (Bonora and Cotta-Ramusino 1983, Quirós *et al* 1981) on the grounds that the vertical part of the connection has no spacetime dependence, and the preferred approach has been to regard the ghost as the Maurer-Cartan form on the (infinite-dimensional) space of gauge transformations, and the BRS operator as the exterior derivative on this space (see also the introductory remarks in Stora (1983)). In this letter we wish to argue again the case for the Thierry-Mieg approach, showing that it leads to an extremely simple and natural framework for understanding anomalies.

It is well known that the correct mathematical setting for describing Yang-Mills fields, for a given gauge group G, is the principal G-bundle (Kobayashi and Nomizu 1963, Spivak 1979). A principal G-bundle is a quadruple $(E, \pi, B, \cdot)$, where E is the total space, π is the projection from E to the base space B and $\cdot : E\times \mathrm{G}\to E, (p, g)\to p\cdot g$ is a right G action on E, satisfying the conditions

(i) $\pi(p\cdot g)=\pi(p)\,\forall p\in E,\ g\in \mathrm{G}$ and $p\cdot(gk)=(p\cdot g)\cdot k\,\forall p\in E,\ g,\ k\in \mathrm{G}$;

(ii) $\forall x\in B$, $\exists$ a neighbourhood U of x and a diffeomorphism $h\colon \pi^{-1}(U)\to U\times \mathrm{G}$ such that $h(p)=(\pi(p), g(p))$ where $g(p)$ satisfies $g(p\cdot k)=g(p)k$.

A local section σ is a continuous map from a neighbourhood U in B to E satisfying $\pi(\sigma(x))=x$, for all x in U. A local section induces local coordinates on E: let $\{x^\mu\}$ be coordinates on U and $\{y^\alpha\}$ be coordinates on G. Then we assign to a point p in $\pi^{-1}(U)$ the coordinates $\{x^\mu, y^\alpha\}$, where $\{x^\mu\}$ are the coordinates of $\pi(p)$ and $\{y^\alpha\}$ are the coordinates of $g(p)$, the unique element of G satisfying $p=\sigma(\pi(p))\cdot g(p)$.

An Ehresmann connection is a g-valued 1-form ω on the total space E (where g denotes the Lie algebra of G) satisfying:

(i) $\omega(\tilde{X})=X$, $\forall x\in g$ ($\tilde{X}$ is the vector field induced by X);

(ii) $\omega(R_{g*}Y)=g^{-1}\omega(Y)g$ for all $g\in \mathrm{G}$ and vectors Y on E (R_{g*} is the action on vectors induced by right multiplication $R_g\colon E\to E$, $R_g(p)=p\cdot g$).

From (i) one can show that the vertical part of ω is the pullback under h of the Maurer-Cartan form $g(y)^{-1}\,\mathrm{d}g(y)$ on G and thus is independent of the x coordinates. This feature has led to the criticism, mentioned in the introduction, of Thierry-Mieg's interpretation of the ghost field as the vertical part of ω.

One can obtain a g-valued 1-form A_σ on a neighbourhood U in the base space by pulling back ω with a local section σ. Clearly A_σ depends on the section chosen. The significance of the two conditions on ω is that all 1-forms obtained in this way are related by gauge transformation: let σ and τ be two sections such that, on the overlap $U_\sigma\cap U_\tau$, $\sigma(x)=\tau(x)\cdot h(x)$. Then

$$A_\tau=h^{-1}A_\sigma h+h^{-1}\,\mathrm{d}h.$$

See Spivak (1979) for a proof and a thorough discussion of this point of view. Thus ω gives rise to Yang-Mills potentials A, being collections of g-valued 1-forms $\{A_i\}$, defined on open sets $\{U_i\}$ covering B, related on the overlaps by gauge transformations.

A vector X on E is said to be horizontal if $\omega(X)=0$. Every vector X on E has a unique decomposition $X=X^{\mathrm{H}}+X^{\mathrm{V}}$, where X^{H} is horizontal and X^{V} is vertical, i.e. $\pi_*(X^{\mathrm{V}})=0$. We call a p-form φ horizontal if $\varphi(X_1,\ldots,X_p)=\varphi(X_1^{\mathrm{H}},\ldots,X_p^{\mathrm{H}})\,\forall X_1,\ldots,X_p$ on E.

We now proceed to show how ω may be decomposed into a part 'along the section', which may be identified with A (we will drop the subscript referring to the section, unless this causes any confusion) and a remainder χ which will be identified with the ghost field. Firstly we choose a coordinate system on E as described above by means of a section, where we make the special choice of a horizontal section σ^{H}, i.e. a section which is such that all its tangent vectors in E are horizontal. These coordinates will be denoted (x, y) (for convenience we drop the indices on x and y) and this particular choice has the advantage that $\omega = \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y = \omega_y\,\mathrm{d}y$ only, as ω has to annihilate all vectors ∂_x. We now choose an arbitrary section σ to pull back ω to A. Let the coordinates of the section σ be given by $(x, s(x))$ parametrised by x in the base. Then we calculate $A = \sigma^*(\omega_y(y)\,\mathrm{d}y) = \omega_y(s(x))s'(x)\,\mathrm{d}x$, where s' is shorthand for $\partial s^\alpha/\partial x^\mu$. Next we change coordinates on E to $(x', y') = (x, y - s(x))$ and express ω in terms of (x', y'): $\omega = \omega_y(y)\,\mathrm{d}y = \omega_y(y' + s(x'))\,\mathrm{d}y' + \omega_y(y' + s(x'))s'(x')\,\mathrm{d}x'$. Clearly on the section, where $y' = 0$, we may identify the second term of this expression with A. The first term $\omega_{y'}\,\mathrm{d}y' = \omega_y(s(x'))\,\mathrm{d}y'$ we will denote by χ, the ghost field. It is important to observe that χ is x'-dependent, avoiding the problem in the Thierry-Mieg formulation. This is related to the fact that the ghost is not 'vertical' (in the sense $\chi = \chi_y\,\mathrm{d}y$ only) as $\mathrm{d}y' = \mathrm{d}y - s'\,\mathrm{d}x$. The point to notice is that one has, *a priori*, two independent sections, one to fix convenient horizontal and vertical coordinates on E, and a separate one to pull back the Yang-Mills connection A.

The key result concerning the Ehresmann connection is the Cartan-Maurer structure equation. Defining the curvature Ω as the horizontal part of $\tilde{\mathrm{d}}\omega$, where $\tilde{\mathrm{d}}$ is the exterior derivative on E, i.e. $\Omega(X, Y) = \tilde{\mathrm{d}}\omega(X^{\mathrm{H}}, Y^{\mathrm{H}})$ for all vectors X, Y on E, this equation states

$$\Omega = \tilde{\mathrm{d}}\omega + \tfrac{1}{2}[\omega, \omega]. \tag{1}$$

Put slightly differently, if instead we define Ω by (1), then we may deduce that Ω is horizontal. In terms of the (x, y) coordinate system defined above, this means $\Omega = \Omega_{xx}\,\mathrm{d}x\,\mathrm{d}x + \Omega_{xy}\,\mathrm{d}x\,\mathrm{d}y + \Omega_{yy}\,\mathrm{d}y\,\mathrm{d}y = \Omega_{xx}\,\mathrm{d}x\,\mathrm{d}x$ only. We now express Ω in the primed coordinates induced by the choice of section σ. Splitting the exterior derivative up as $\tilde{\mathrm{d}} = \mathrm{d}x'\,\partial_{x'} + \mathrm{d}y'\,\partial_{y'} \equiv \mathrm{d} + s$, we have

$$\Omega_{x'x'}\,\mathrm{d}x'\,\mathrm{d}x' = \mathrm{d}A + \tfrac{1}{2}[A, A] \tag{2}$$

$$\Omega_{x'y'}\,\mathrm{d}x'\,\mathrm{d}y' = sA + \mathrm{d}\chi + [A, \chi] \tag{3}$$

$$\Omega_{y'y'}\,\mathrm{d}y'\,\mathrm{d}y' = s\chi + \tfrac{1}{2}[\chi, \chi]. \tag{4}$$

On the other hand, being a 2-form, Ω transforms as a tensor and hence

$$\Omega_{x'x'} = \Omega_{xx} + s'\Omega_{xy} + s's'\Omega_{yy}$$

$$\Omega_{x'y'} = \Omega_{xy} + s'\Omega_{yy}$$

$$\Omega_{y'y'} = \Omega_{yy}.$$

Thus $\Omega_{x'y'} = \Omega_{y'y'} = 0$, the right-hand sides of (3) and (4) vanish and we regain Thierry-Mieg's interpretation of the BRS operator s as part of the exterior derivative on the bundle (in this improved version s is not 'along the fibre' but 'along the coordinates y''). The non-vanishing part of Ω, $\Omega_{x'x'}\,\mathrm{d}x'\,\mathrm{d}x'$, may be naturally identified with the Yang-Mills fields strength 2-form $F = \sigma^*\Omega$ on the base space.

In conclusion, we have shown how a slightly modified Thierry-Mieg approach leads to a natural geometric interpretation of the ghost field and the BRS operator of quantum field theory. We now apply this formalism to clarify the cohomological significance of anomalies.

When a classical field theory possesses a symmetry, which fails to hold after (some of) the fields are quantised, we speak of an anomaly. An example of a model where this can occur is the theory of Yang–Mills gauge fields, with group G, coupled gauge-invariantly to chiral fermions. The gauge invariance of the classical action $I^{\text{cl}}[A, \psi]$ is lost when the fermions are quantised and integrated out leaving an effective action $W[A]$. The statement of the anomaly is then

$$sW[A] = \int_B \mathrm{G}(\chi, A) \neq 0 \tag{5}$$

where $sW[A]$ stands for the functional variation of W under an infinitesimal gauge transformation $sA = -\mathrm{d}\chi - [A, \chi]$. The right-hand side (the integrated anomaly) is a local functional of χ and A, polynomial in the fields and linear in χ, whereas the effective action is non-local in A.

The Wess–Zumino consistency condition is obtained by applying s to both sides of (5) and using the fact that $s^2 = 0$:

$$s \int_B \mathrm{G}(\chi, A) = 0. \tag{6}$$

The integrated anomaly is thus closed under the BRS operator s; if however it is locally s-exact, i.e. it equals s applied to a local functional, then the anomaly can be removed by adding a local counterterm to the effective action $W[A]$. Thus for non-trivial anomalies we require

$$\int_B \mathrm{G}(\chi, A) \neq s \int_B C(A). \tag{7}$$

Because of (6) and (7) a non-trivial integrated anomaly represents an s-cocycle within the restricted class of local functionals. This is the local cohomology introduced by Bonora and Cotta-Ramusino (1983).

The integrand $\mathrm{G}(\chi, A)$ is only defined up to the addition of an arbitrary d-exact form, if we make the usual assumption that B is without boundary, or that all fields vanish at infinity. At this level (6) and (7) take the form

$$s\mathrm{G}(\chi, A) + \mathrm{d}H(\chi, A) = 0 \tag{8}$$

$$\mathrm{G}(\chi, A) \neq \mathrm{d}K(\chi, A) + sC(A). \tag{9}$$

If spacetime is assumed to have four dimensions, solutions to (8) and (9) may be obtained by finding 5-forms $\mathrm{G}_5(\omega, \Omega)$ on E, polynomial in ω and Ω, which are non-trivial $\tilde{\mathrm{d}}$-cocycles. This is seen by expressing $\mathrm{G}_5(\omega, \Omega)$ in terms of the (x', y') basis described above and expanding 'in powers of χ'

$$\mathrm{G}_5(\omega, \Omega) = \mathrm{G}_5(A + \chi, \Omega) = \mathrm{G}_4^1(A, \chi, \Omega) + \mathrm{G}_3^2(A, \chi, \Omega) + \ldots + \mathrm{G}_0^5(\chi) \tag{10}$$

where the lower index refers to the $\mathrm{d}x'$ form degree (spacetime form degree) and the upper index refers to the $\mathrm{d}y'$ form degree (ghost number). Equations (8) and (9) are regained on imposing the non-trivial cocycle condition

$$\tilde{\mathrm{d}}\mathrm{G}_5 = (\mathrm{d} + s)\mathrm{G}_5 = 0 \tag{11}$$

$$\mathrm{G}_5 \neq \tilde{\mathrm{d}}\mathrm{G}_4 \tag{12}$$

and identifying G_4^1 in the expansion of G_5 with the anomaly G.

A solution to (11) and (12) is provided by the Chern–Simons form $Q_5(\omega, \Omega)$ defined by (see Eguchi *et al* 1980)

$$Q_5 = 3 \int_0^1 P(\omega, \Omega_t, \Omega_t)\, dt \tag{13}$$

where $\omega_t = t\omega$, $\Omega_t = t\, d\omega + \frac{1}{2}t^2[\omega, \omega]$ and $P(\cdot, \cdot, \cdot)$ is an invariant, symmetric, trilinear form on g. This is because $\tilde{d}Q = P(\Omega, \Omega, \Omega)$ vanishes, being a horizontal 6-form, as there are only four horizontal dimensions available (Thierry-Mieg 1984, 1985). Furthermore, Q_5 is not $\tilde{d}$-exact, because the term of maximal ghost number in the expansion of Q_5, namely

$$Q_0^5 = P(\chi, [\chi, \chi], [\chi, \chi]) \tag{14}$$

would have to equal sG_0^4 for some ghostly 4-form G_0^4. The only candidate for G_0^4 is

$$P(\chi, \chi, [\chi, \chi])$$

which vanishes identically due to symmetrisation/antisymmetrisation.

It should be pointed out that the above is not in disagreement with Dubois-Violette *et al* (1985a, b) where it was found that the $\tilde{d}$ cohomology is trivial. The point is that they deal with a differential algebra without any further horizontality or dimensionality considerations. Thus in their formalism $\tilde{d}Q_5 = P(\Omega, \Omega, \Omega) \neq 0$ and therefore Q_5, not being closed, cannot be a cocycle.

Secondly we remark that, for the more general case of 'anomalies' with ghost number greater than one, and for arbitrary spacetime dimension, there may be several solutions to (11) and (12), obtained by forming products of Chern–Simons forms and invariant polynomials of Ω (Thierry-Mieg 1984). In our case, if we assume that G is simple, Q_5 is the only solution, because combinations like $Q_3(\omega, \Omega)P_2(\Omega)$ and $Q_1(\omega)P_4(\Omega, \Omega)$ necessarily vanish.

We now ask how this non-trivial de Rham cohomology $H^5(E) \neq 0$ arises. A hint is provided by noticing that E is the (twisted) product of B and G, and that $H^5(\mathrm{G}) \neq 0$ if G admits a non-vanishing invariant, symmetric, trilinear form $P(\cdot, \cdot, \cdot)$. This last fact may be proved directly or can be established by comparing a table of the orders of non-trivial Casimir elements for the simple groups (see e.g. Cvitanović 1984), which are in one-to-one correspondence with invariant, symmetric, multilinear forms, and a table of the cohomology of the simple groups (Iyanaga and Kawada 1977): each order n Casimir element corresponds to non-trivial $(2n-1)$ cohomology.

A direct way to establish this connection with $H^5(\mathrm{G})$ is to go to a gauge where $A = 0$, by using a horizontal section to pull back ω. Then $\omega = \omega_y(y)\, dy = \chi = g^{-1}\, dg$ and $Q_5(\omega, \Omega) = Q_5(\chi) = P(\chi, [\chi, \chi], [\chi, \chi])$, which represents, as we have seen before in equation (14), non-trivial cohomology on G, being a cocycle with respect to s, the exterior derivative on G.

Thus one achieves a rather roundabout link with the 'index theorem approach' to gauge anomalies, where the anomaly is related to the topology $\Pi_5(\mathrm{G}) \supset \mathbb{Z}$ (Sumitani 1984, Alvarez-Gaumé and Ginsparg 1984). A quick survey of the homotopy groups of simple Lie groups (Iyanaga and Kawada 1977) shows that this is equivalent to $H^5(\mathrm{G}) \neq 0$.

In conclusion, the method we have described shows clearly the direct relationship between gauge anomalies and the cohomology of G, whilst at the same time providing a natural geometric interpretation for the ghost and the BRS operator, without the drawback of the Thierry-Mieg (1980a, b) approach, that the ghost is not spacetime dependent. The topology of G also enters in a crucial way in the 'index theorem approach', which possesses the advantage that it derives the correct normalisation as well as the functional form of anomalies. This approach however requires much more complicated mathematical machinery.

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